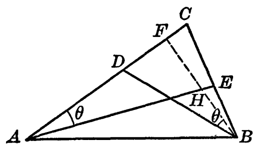
Solution by the Proposer.

Let C be the third vertex. By hypothesis $CAE < CBD$ and $BAE < ABD$. A point F can then be found on CD such that $DBF = CAE$. Let BF cut AE in H. AHF and BDF are similar, having equal angles at A and B and the same angle at F. Therefore,



$$AH : BD = AF : BF. \tag{1}$$

Also, since $ABD > BAE$, $ABF > BAF$. Consequently,

$$AF > BF. \tag{2}$$

From (1) and (2), it follows that $AH > BD$. Therefore, $AE > AH > BD$, which was to be proved.

Corollary: *If AE and BD are equal and divide their angles in the same ratio, the triangle is isosceles.*

For, if the angles A and B were not equal the parts of one would be respectively less than the corresponding parts of the other and AE and BD would be unequal, which is contrary to hypothesis.

In particular, if the bisectors of two angles of a triangle are equal, the triangle is isosceles.

2736 [December, 1918]. Proposed by M. COHEN, Freshman, Johns Hopkins University.

Prove by elementary geometry that the orthocenter, the centroid, and the circumcenter of a triangle lie on a line (the Euler line), and that the centroid lies between the other two and is twice as far from the orthocenter as from the circumcenter.

Solution by J. L. Riley, Stephenville, Texas.

Let ABC be the triangle under consideration; O and G the circumcenter and centroid, BE and CF perpendicular, respectively, to AC and AB. Let mid-point of AC be B'.

Produce OG to meet the altitude BE at K. The triangles OGB' and KGB are similar, for OB' is parallel to BK, since each is perpendicular to AC. Then $OG : GK = B'G : GB = 1 : 2$ and hence, $GK = 2\,OG$.

If OG is produced to meet the altitude CF at K', it follows in the same way that $GK' = 2\,OG$. Therefore, $GK' = GK$ and K' coincides with K. Hence BE and CF meet at K and K is the orthocenter. Hence, circumcenter, centroid, and orthocenter lie on the same line.

Also solved by H. L. Olson, C. P. Sousley, and the Proposer.

2737 [January, 1919]. Proposed by C. N. SCHMALL, New York City.

Employing Maclaurin's theorem, or otherwise, expand the following three functions (1) $e^{\tan^{-1}x}$ as far as x^6; (2) $e^{\sin x}$ as far as x^8; and (3) $\tan x$ as far as x^9.

Solution by Elmer Latshaw, West Philadelphia, Pennsylvania.

The successive differentiation required by Maclaurin's theorem in the development of the given functions is long and laborious, but the required developments may be obtained by comparing the derivative of the function with the function itself.

Assume

$$e^{\tan^{-1}x} = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_6x^6 + \cdots. \tag{1}$$

Differentiating both sides of (1),

$$\begin{aligned} e^{\tan^{-1}x}\frac{1}{1+x^2} &= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + 6a_6x^5 + \cdots \\ &= (a_0 + a_1x + a_2x^2 + \cdots a_6x^6 + \cdots)(1 - x^2 + x^4 - x^6 + \cdots). \end{aligned}$$

Equating coefficients of like powers of x,

$$a_1 = a_0,\quad 2a_2 = a_1,\quad 3a_3 = a_2 - a_0,\quad 4a_4 = a_3 - a_1,\quad 5a_5 = a_4 - a_2 + a_0,\quad 6a_6 = a_5 - a_3 + a_1.$$

Equation (1) by making $x = 0$ gives $a_0 = 1$ and the preceding equations give

$$a_1 = 1,\qquad a_2 = \tfrac{1}{2},\qquad a_3 = -\tfrac{1}{6},\qquad a_4 = -\tfrac{7}{24},\qquad a_5 = \tfrac{1}{24},\qquad a_6 = \tfrac{29}{144}.$$

Hence,

$$e^{\tan^{-1}x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{7x^4}{24} + \frac{x^5}{24} + \frac{29x^6}{144} - \cdots.$$

$e^{\sin x}$ may be similarly developed.

$$e^{\sin x} = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_8x^8 + \cdots. \tag{2}$$

Differentiating,

$$e^{\sin x}\cos x = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + 8a_8x^7 + \cdots$$
$$= (a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_8x^8 + \cdots)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots\right).$$

Equating coefficients,

$$a_1 = a_0, \quad 2a_2 = a_1, \quad 3a_3 = a_2 - \frac{a_0}{2!}, \quad 4a_4 = a_3 - \frac{a_1}{2!}, \quad 5a_5 = a_4 - \frac{a_2}{2!} + \frac{a_0}{4!},$$

$$6a_6 = a_5 - \frac{a_3}{2!} + \frac{a_1}{4!}, \quad 7a_7 = a_6 - \frac{a_4}{2!} + \frac{a_2}{4!} - \frac{a_0}{6!}, \quad 8a_8 = a_7 - \frac{a_5}{2!} + \frac{a_3}{4!} - \frac{a_1}{6!}.$$

Making $x = 0$ in equation (2) gives $a_0 = 1$ and the preceding equations give

$a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = 0, \quad a_4 = -\frac{1}{8}, \quad a_5 = -\frac{1}{15}, \quad a_6 = -\frac{1}{240}, \quad a_7 = \frac{1}{90}, \quad a_8 = \frac{31}{5760}.$

Hence,
$$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^5}{15} - \frac{x^6}{240} + \frac{x^7}{90} + \frac{31x^8}{5760} + \cdots.$$

To develop $\tan x$, we notice that $\tan(-x) = -\tan x$. Hence, the expansion will contain only odd powers of x.

Assume
$$\tan x = a_1x + a_3x^3 + a_5x^5 + a_7x^7 + a_9x^9 + \cdots.$$

Differentiating,

$$\sec^2 x = a_1 + 3a_3x^2 + 5a_5x^4 + 7a_7x^6 + 9a_9x^8 + \cdots = 1 + \tan^2 x$$
$$= 1 + (a_1x + a_3x^3 + a_5x^5 + a_7x^7 + a_9x^9 + \cdots)^2.$$

Equating coefficients of like powers of x,

$$a_1 = 1, \quad 3a_3 = a_1^2, \quad 5a_5 = 2a_1a_3, \quad 7a_7 = 2a_1a_5 + a_3^2, \quad 9a_9 = 2a_1a_7 + 2a_3a_5.$$

From these, we obtain

$$a_1 = 1, \quad a_3 = \tfrac{1}{3}, \quad a_5 = \tfrac{2}{15}, \quad a_7 = \tfrac{17}{315}, \quad a_9 = \tfrac{62}{2835}.$$

Hence,
$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots.$$

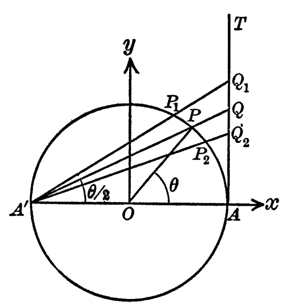
Also solved by E. D. Grant, H. L. Olson, J. L. Riley, and the Proposer.

2738 [January, 1919]. Proposed by W. D. CAIRNS, Oberlin College.

Prove that between any two points on a unit circle with its center at the origin there is a point whose coördinates are rational.

Solution by P. J. Daniell, Rice Institute.

Let $A'OA$ be the diameter on the x-axis, and let P_1, P_2 be the two given points. Through A draw AT perpendicular to $A'OA$. Let $A'P_1$, $A'P_2$ intersect AT in Q_1, Q_2. By the theory of irrational numbers between the points Q_1, Q_2 on AT there is a point Q such that AQ is rational and indeed equal to $2m/n$, where m, n are integers. Let $A'Q$ intersect the circle in P. Then P is the required point. It is assumed, and this involves no loss in generality, that P_1, P_2 lie on the same side of the x-axis. Then P lies between P_1 and P_2.



$$x = \cos\theta = \frac{1 - \tan^2\dfrac{\theta}{2}}{1 + \tan^2\dfrac{\theta}{2}} = \frac{1 - \dfrac{AQ^2}{A'A^2}}{1 + \dfrac{AQ^2}{A'A^2}} = \frac{1 - \dfrac{m^2}{n^2}}{1 + \dfrac{m^2}{n^2}} = \frac{n^2 - m^2}{n^2 + m^2}$$

is rational. Similarly $y = \dfrac{2mn}{n^2 + m^2}$ is rational.

Also solved by R. A. Johnson, H. L. Olson, A. Pelletier, W. R. Ransom, J. Rosenbaum, E. Swift, and the Proposer.